1. $m^*(A) \stackrel{def}{=} \inf\left\{\sum_{n\in\mathbb{N}} l(I_n) : \text{COIC } \{I_n\}_{n\in\mathbb{N}} \text{ of } A\right\}$ (def α)

$= \inf\{m^*(G) : \text{open } G \supseteq A\}$ ($\stackrel{def}{=} \beta$)

Let $\{I_n : n\in\mathbb{N}\}$ be an open cover of A and $G_0 := \bigcup_{n=1}^{\infty} I_n$ (open) $G \supseteq A$ so

$\beta \leq m^*(G_0) \leq \sum_{n\in\mathbb{N}} l(I_n)$ (by def of $m^*(G_0)$ as $\{I_n\}$ is COIC of G_0)

This implies that $\beta \leq m^*(A)$ (def of $m^*(A)$)

That $\beta \geq m^*(A)$ is due to the monotonicity of m^*.

2. $A \triangle B$ is of outer mea $=$ zero $\Rightarrow m^*(A) = m^*(B)$

pf. $m^*(A) = m^*(A\cup B) = m^*(B)$

since $(A\cup B)\setminus A$ is of outer mea. zero

3. $E \in \mathfrak{M}$ iff

(*) $m^*(A) = m^*(A\cap E) + m^*(A\cap \tilde{E})$

equivalently iff

(**) $m^*(A) \geq m^*(A\cap E) + m^*(A\cap \tilde{E})$ whenever $m^*(A) < +\infty$

Ex. 1) $m^*(E) = 0 \Rightarrow E \in \mathfrak{M}$

2) $m^*(E_1 \triangle E_2) = 0,\ E_1 \in \mathfrak{M} \Rightarrow E_2 \in \mathfrak{M}$.

(sometime, say $E_1 \sim E_2$)

Pf $m^*(A\cap(E_1\cup E_2))$

$$\underset{(E_1\in\mathfrak{M})}{=} m^*\Big((A\cap(E_1\cup E_2))\cap E_2\Big) + m^*\Big((A\cap(E_1\cup E_2))\cap \tilde{E}_2\Big)$$

$$\underset{E_1\cap E_2=\emptyset}{=} m^*(A\cap E_2) + m^*(A\cap E_1)$$

and, more generally, $\forall n\in\mathbb{N}\setminus\{1\}$, $\{E_i : i=1,\cdots,n\}\subseteq\mathfrak{M}$, disjoint

$$m^*(A\cap(E_1\cup E_2\cup\cdots\cup E_n)) = \sum_{i=1}^{n} m^*(A\cap E_i)$$

because you can prove by MI with

$$LHS = m^*\Big(A\cap\bigcup_{i=1}^{n} E_i\cap E_n\Big) + m^*\Big(A\cap\bigcup_{i=1}^{n} E_i\cap\tilde{E}_n\Big)$$

$$= m^*(A\cap E_2) + m^*\Big(A\cap\bigcup_{i=1}^{n-1} E_i\Big)$$

Prop 3. Let $\mathcal{A}$ be an algebra of sets. Then it is an σ-alg iff it is stable w.r.t. countable disjoint unions:

$$A_i\cap A_j=\emptyset\ \forall i\neq j \ \&\ A_i\in\mathcal{A}\ \forall i\in\mathbb{N}.$$

$$\Rightarrow \bigcup_{i\in\mathbb{N}} A_i\in\mathcal{A}$$

Hint: $\bigcup_{i\in\mathbb{N}} A_i = A_1\cup(A_2\setminus A_1)\cup\Big(A_3\setminus\bigcup_{i=1}^{2} A_i\Big)\cup\cdots$

$\overset{\text{i.e.}}{=}\bigcup_{j\in\mathbb{N}} B_j$ where each $B_j = A_j\setminus\Big(\bigcup_{i<j} A_i\Big)$

Th 2. $\mathfrak{M}$ is an σ-alg & $m := m^*|_{\mathfrak{M}}$ is a measure.

~~Prop1~~ Th 1. $\mathcal{M}$ is an σ-alg. (Already know: $E \in \mathcal{M} \Leftrightarrow \tilde{E} \in \mathcal{M}$)

Pf. Let $E = E_1 \cup E_2$ with $E_1, E_2 \in \mathcal{M}$.

Let $A \subseteq \mathbb{R}$. Then

$$
\begin{aligned}
m^*(A) &= m^*(A \cap E_1) + m^*(A \cap \tilde{E}_1) \\
&= m^*(A \cap E_1) + \left[m^*(A \cap \tilde{E}_1 \cap E_2) + m^*(A \cap \tilde{E}_1 \cap \tilde{E}_2) \right] \\
&\geq m^*\left((A \cap E_1) \cup (A \cap \tilde{E}_1 \cap E_2) \right) + m^*\left(A \cap \widetilde{(E_1 \cup E_2)} \right) \\
&= m^*\left(A \cap (E_1 \cup E_2) \right) + m^*\left(A \cap \widetilde{(E_1 \cup E_2)} \right)
\end{aligned}
$$

resp., by $E_1 \in \mathcal{M}$, $E_2 \in \mathcal{M}$, subadditivity and

$$E_1 \cup E_2 = E_1 \cup (\tilde{E}_1 \cap E_2) \quad (= E_1 \cup (E_2 \setminus E_1))$$

$$\Rightarrow A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap \tilde{E}_1 \cap E_2).$$

Ex1. $E_1, E_2 \in \mathcal{M} \Rightarrow E_1 \setminus E_2 \in \mathcal{M}$

Ex2. If $E = \bigcup_{i=1}^{n} E_i$ with each $E_i \in \mathcal{M}$

$\Rightarrow E = \bigcup_{i=1}^{n} F_i$ with each $F_i \in \mathcal{M}$ & $F_i \cap F_j = \emptyset$ $\forall i \neq j$

Ex3. Extend Ex 2 to countable case.

Prop2: $m^*(A \cap (E_1 \cup E_2)) = m^*(A \cap E_1) + m^*(A \cap E_2)$

if $E_1 \cap E_2 = \emptyset$, $E_1, E_2 \in \mathcal{M}$ and $A \subseteq \mathbb{R}$

Note. When $E_2 = \tilde{E}_1$, the above eq is same as (*) the def. of $E_1 \in \mathcal{M}$